Assignment 13

This homework is due *Thursday* Dec 10.

There are total 20 points in this assignment. 17 points is considered 100%. If you go over 17 points, you will get over 100% for this homework and it will count towards your course grade (not over 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much. This assignment covers sections 6.4, 7.1, 7.2 in Bartle–Sherbert.

(1) (3pt) Apply Newton's Method to find recursive relation for approximating $\sqrt[3]{2}$. Find an interval $I^* \subseteq I = [1, 2]$ s.t. convergence of a sequence given by the above relation and $x_1 \in I^*$ is guaranteed.

1. Riemann Integral

(2) [1pt] (Part of 7.1.1) If $I = [0, 4]$, calculate the norms of the following partitions:
   (a) $\mathcal{P}_1 = (0, 1, 2, 4)$,
   (b) $\mathcal{P}_2 = (0, 2, 3, 4)$,
   (c) $\mathcal{P}_3 = (0, 1, 1.5, 2, 3.4, 4)$.

(3) (Part of 7.1.2) If $f(x) = x^2$ for $x \in [0, 4]$, calculate the following Riemann sums, where $\dot{\mathcal{P}}_i$ has the same partition points as in the previous problem, and the tags are selected as indicated.
   (a) [1pt] $\mathcal{P}_1$ with the tags at the left endpoints of the subintervals.
   (b) [1pt] $\mathcal{P}_2$ with the tags at the right endpoints of the subintervals.

(4) [2pt] (7.1.8) If $f \in \mathcal{R}[a, b]$ and $|f(x)| \le M$ for all $x \in [a, b]$, show that

$$\left| \int_a^b f \right| \le M(b - a)$$

by directly inspecting the Riemann sums.

(5) [3pt] (Theorem 5.4.3) In this problem we prove a theorem that we need below to show integrability of continuous functions.
Uniform Continuity Theorem. Let $I = [a, b]$ be a closed bounded interval and $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I, that is for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, x' \in I$ if $|x - x'| < \delta$ then $|f(x) - f(x')| < \varepsilon$.
   Follow the steps:
   (a) Suppose uniform continuity fails for some $\varepsilon_0 > 0$. Then, in particular, for every $\delta_n = 1/n$, there are points x_n, x'_n that break the uniform continuity condition for given $\varepsilon = \varepsilon_0$ and $\delta = \delta_n$. Write explicitly what it means in terms of inequalities involving $|x_n - x'_n|$ and $|f(x_n) - f(x'_n)|$.

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(b) Show that (x_n) has a convergent subsequence $(x_{n_k}) \to z$. Show that $z \in I$.

(c) Use continuity of f at the above z to obtain a contradiction with choice of (x_n), (x'_n). (Start by finding δ such that $|f(x) - f(z)| < \varepsilon_0/2$ whenever $|x - z| < \delta$ and $x \in I$.)

(6) [3pt] Let $f : [a, b] \to \mathbb{R}$ and let $\mathcal{P}, \mathcal{Q}$ be partitions of $[a, b]$. Suppose $\mathcal{Q}$ is a *refinement* of $\mathcal{P}$, i.e. every point of $\mathcal{P}$ is also a point of $\mathcal{Q}$ (for example, $\mathcal{P}_3$ in Problem 2 is a refinement of $\mathcal{P}_1$). Further, suppose that f is such that $|f(x) - f(x')| < \varepsilon$ whenever $|x - y| \le \|\mathcal{P}\|$. Following the steps below, show that

$$|S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})| < \varepsilon \cdot (b - a)$$

for any choice of tags.

(a) Let $\mathcal{P} = \{x_0 = a, x_1, \ldots, x_n = b\}$. Fix an m between 1 and n and let $y_{k-1}, y_k, \ldots, y_\ell$ be the points of $\mathcal{Q}$ on the interval $[x_{m-1}, x_m]$:

$$x_{m-1} = y_{k-1} < y_k < \ldots < y_{\ell-1} < y_\ell = x_m.$$

Let t_m be the tag of $[x_{m-1}, x_m]$, and let $s_k, s_{k+1}, \ldots, s_\ell$ be tags of $[y_{k-1}, y_k]$, $\ldots$, $[y_{\ell-1}, y_\ell]$.
Show that

$$\left| (x_m - x_{m-1}) f(t_m) - \sum_{i=k}^{\ell} (y_i - y_{i-1}) f(s_i) \right| < (x_m - x_{m-1}) \cdot \varepsilon.$$

(Start by observing that $(x_m - x_{m-1}) = \sum_{i=k}^{\ell} (y_i - y_{i-1})$.)

(b) Use the triangle inequality and the above inequality to get the desired inequality.

(7) [6pt] (Theorem 7.2.7) Prove the theorem by following the steps below.
Theorem. If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ then $f \in R[a, b]$.

(a) Define a partition $\dot{\mathcal{P}}_n$ to split $[a, b]$ into 2^n equal parts, $x_0 = a$, $x_1 = a + \frac{b-a}{2^n}$, and so on (with any choice of tags). Use uniform continuity of f on $[a, b]$ (provided by Problem 5) and Problem 6 to show that the sequence $(S(f, \dot{\mathcal{P}}_n))$ is Cauchy. (Given an $\varepsilon > 0$, start by picking n so that $|f(x) - f(x')| < \varepsilon/(b - a)$ whenever $|x - x'| \le (b - a)/2^n$.) Denote the limit of this sequence by L.

(b) Given an $\varepsilon > 0$, for an arbitrary tagged partition $\dot{\mathcal{Q}}$, consider the partition $\mathcal{R} = \mathcal{Q} \cup \mathcal{P}_n$ (that is, $\mathcal{R}$ consists of points of $\mathcal{Q}$ and $\mathcal{P}_n$ together). Note that $\mathcal{R}$ is a refinement of both $\mathcal{Q}$ and $\mathcal{P}_n$ to argue by Problem 6 that if $\|\mathcal{Q}\|$ and $\|\mathcal{P}_n\|$ are small enough, then

$$|S(f, \dot{\mathcal{R}}) - S(f, \dot{\mathcal{Q}})| < \varepsilon \text{ and } |S(f, \dot{\mathcal{R}}) - S(f, \dot{\mathcal{P}}_n)| < \varepsilon.$$

(c) Rewrite

$$S(f, \dot{\mathcal{Q}}) - L = \Big(S(f, \dot{\mathcal{Q}}) - S(f, \dot{\mathcal{R}})\Big) + \Big(S(f, \dot{\mathcal{R}}) - S(f, \dot{\mathcal{P}}_n)\Big) + \Big(S(f, \dot{\mathcal{P}}_n) - L\Big)$$

to show that $|S(f, \dot{\mathcal{Q}}) - L| < \varepsilon$ if $\|\dot{\mathcal{Q}}\|$ is small enough, completing the proof.

(d) Congratulate yourself.